Chapter 3 notes   Name Key (2014)

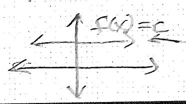
need flashcards for this chapter!

3.3 Rules for Differentiation

Derivative = rate of change = slope

Rule 1) *derivative of a constant*

If $f(x)=c$, then $f'(x)=0$



slope is zero

Rule 2) *power rule*, $x \neq 0$

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

examples

Example:

$f(x)$	x^1	x^2	x^3	x^4	$x^{2/3}$	$1/x$ $=x^{-1}$	$1/x^2$ $=x^{-2}$
$f'(x)$	$1 \cdot x^0$ $=1$	$2 \cdot x^1$ $=2x$	$3 \cdot x^2$ $=3x^2$	$4 \cdot x^3$ $=4x^3$	$\frac{2}{3} \cdot x^{-1/3}$ $=\frac{2}{3x^{1/3}}$	$-1x^{-2}$ $=\frac{-1}{x^2}$	$-2x^{-3}$ $=\frac{-2}{x^3}$

Rule 3) *constant multiple rule*

$$\frac{d}{dx}(c \cdot u) = c \cdot \frac{d}{dx}(u)$$

- u is a differentiable function
- c is a constant

constant can move outside

Example: Find the derivative.

$f(x) = 7x^3$

$f' = 7 \cdot \frac{d}{dx}(x^3)$

$= 7 \cdot 3x^2 = \boxed{21x^2}$

$f(t) = \frac{4t^2}{5}$

$f' = \frac{4}{5}\frac{d}{dt}(t^2) = \frac{4}{5} \cdot 2t$

$= \boxed{\frac{8}{5}t}$

$f(x) = 3x^{\frac{1}{2}}$

$f' = 3 \cdot \frac{d}{dx}(x^{1/2})$

$= 3 \cdot \frac{1}{2}x^{-1/2}$

$= \boxed{\frac{3}{2x^{1/2}}}$

$f(x) = 4x^{\frac{-3}{4}}$

$f' = 4 \cdot \frac{d}{dx}(x^{-3/4})$

$= 4 \cdot \frac{-3}{4}x^{-7/4}$

$= \boxed{\frac{-3}{x^{7/4}}}$

Rule 4) *sum and difference rule*

$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

u & v are functions

derivative of sum = sum of derivatives

Example: Find the derivative.

$f(x) = 7x^4 + 3x$

$\frac{d}{dx}(7x^4 + 3x) = \frac{d}{dx}(7x^4) + \frac{d}{dx}(3x)$

$= 4 \cdot 7x^3 + 1 \cdot 3x^0$

$= 28x^3 + 3$

$\boxed{f'(x) = 28x^3 + 3}$

$f(t) = 2t^3 - 7x^0$

$f'(t) = 3 \cdot 2t^2 - 0$

$\boxed{f'(t) = 6t^2}$

Rule 5) *product rule*

$\frac{d}{dx}(u \cdot v) = u \cdot \frac{d}{dx}(v) + v \cdot \frac{d}{dx}(u)$

{Left · d(Right) + Right · d(Left)}

Example: Find the derivative. use product rule

$f(x) = x^2(x + 5)$

$f' = x^2 \cdot \frac{d}{dx}(x+5) + (x+5) \cdot \frac{d}{dx}(x^2)$

$= x^2(1+0) + (x+5)(2x)$

$= x^2 + 2x^2 + 10x$

$\boxed{f' = 3x^2 + 10x}$

$f(a) = (3a + 2a^2)(5 + 4a)$

$f' = (3a + 2a^2)\frac{d}{da}(5+4a) + (5+4a) \cdot \frac{d}{da}(3a+2a^2)$

$= (3a+2a^2)(0+4) + (5+4a)(3+4a)$

$= 12a + 8a^2 + 15 + 20a + 12a + 16a^2$

$\boxed{f' = 24a^2 + 44a + 15}$

Rule 6) *quotient rule*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{d}{dx}(u) - u\cdot\frac{d}{dx}(v)}{v^2}$$

$$\frac{\text{low}\cdot d(\text{high}) - \text{high}\cdot d(\text{low})}{(\text{low})^2}$$

Example: Find the derivative.

"high"

"low"

$$f(t) = \frac{t+1}{3t^2}$$

$$f'(t) = \frac{3t^2\cdot\frac{d}{dt}(t+1) - (t+1)\cdot\frac{d}{dt}(3t^2)}{(3t^2)^2}$$

$$= \frac{3t^2(1+0) - (t+1)(6t)}{9t^4}$$

$$= \frac{3t^2 - (6t^2+6t)}{9t^4}$$

$$= \frac{-3t^2-6t}{9t^4} = \frac{-\cancel{3t}(t+2)}{\cancel{3t}(3t^3)}$$

$$\boxed{f' = \frac{-(t+2)}{3t^3}}$$

might be easier if we simplify first

$$f(x) = \frac{2-\frac{1}{x}}{x+5} = \frac{\frac{2x-1}{x}}{\frac{x+5}{1}} = \frac{2x-1}{x}\cdot\frac{1}{x+5}$$

$$f(x) = \frac{2x-1}{x^2+5x}$$

$$f' = \frac{(x^2+5x)(2) - (2x-1)(2x+5)}{(x^2+5x)^2}$$

$$= \frac{2x^2+10x-(4x^2+10x-2x-5)}{(x^2+5x)^2}$$

$$\boxed{f' = \frac{-2x^2+2x+5}{(x^2+5x)^2}}$$

recall: defn of derivative

$$g' = \lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$$

Example: Given: $g'(x) = \lim_{h\to 0}\frac{\overbrace{(1+h)^2-(1+h)}^{g(a+h)}-\overbrace{g(1)}^{g(a)}}{h}$

$g(x) = x^2 - x$

$g'(x) = 2x-1$

$g(1) = 1^2 - 1 = 1 - 1 = 0 \rightarrow pt(1,0)$

$g'(1) = 2(1)-1 = 2-1 = 1 \rightarrow m=1$

Equation of tangent line of $g(x)$ at $x = 1$: $\boxed{y-0 = 1(x-1)}$

f"(x) or y" = the ***second derivative***

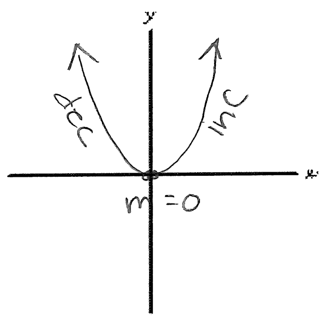
= derivative of first derivative

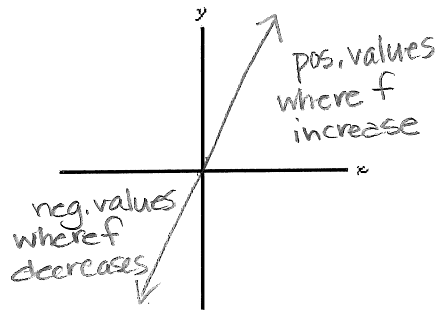
$f(x)=x^2$	$f'(x)$	$f''(x)$

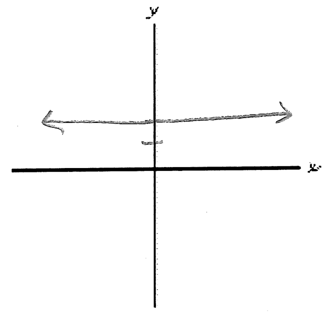
$f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$







$f(x)$	$f'(x)$	$f''(x)$
f increasing ↙↗	$f' > 0$	—
f decreasing ↖↘	$f' < 0$	—
f max (inc, dec)	$f'=0$ or $f'=$ und and changes + to −	$f'' > 0$
f min (dec, inc)	$f'=0$ or $f'=$ und and changes − to +	$f'' < 0$
f concave up	f' is increasing	$f'' > 0$
f concave down	f' is decreasing	$f'' < 0$
f inflection pt	f' min or max	$f''=0$ or $f''=$ und and changes sign

$f(x) = x^3 + x^2 - 2x$

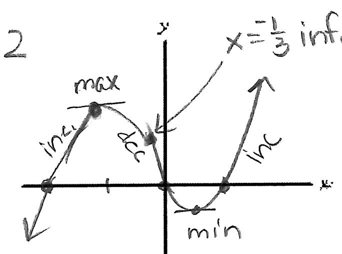
$f'(x) = 3x^2 + 2x - 2$

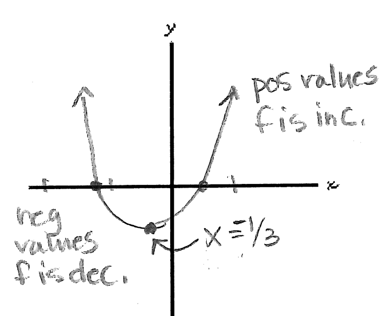
$f''(x) = 6x + 2$

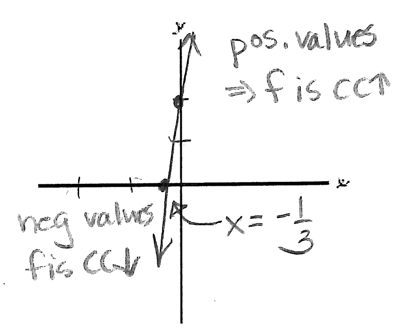
$f = x(x^2 + x - 2)$

$f = x(x+2)(x-1)$

f(x)	f'(x)	f''(x)



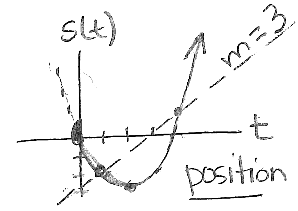




$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(-2)}}{2(3)}$$

$x \approx .5, -1.2$

3.4 Velocity & Other Rates of Change



Displacement: change in position (Δs)

$$\text{disp} = \underbrace{s(t_1)}_{\text{end position}} - \underbrace{s(t_0)}_{\text{start position}}$$

Example: The equation describing the position of an object is $s(t) = (t-2)^2 - 3$ where s is in meters and t is in seconds.

a) Find the displacement of the object from $t = 1$ to $t = 6$.

$$\text{disp} = s(6) - s(1)$$
$$= (6-2)^2 - 3 - [(1-2)^2 - 3] = 4^2 - 3 - [1^2 - 3] = 16 - 3 - 1 + 3 = \boxed{15m}$$

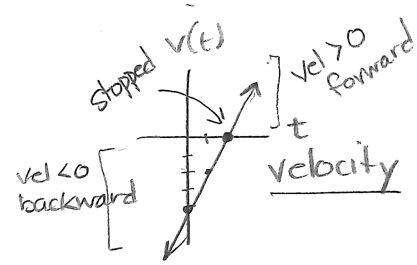
Velocity: rate of change of position

average velocity $= \dfrac{s(t_1) - s(t_0)}{t_1 - t_0}$ "algebra" slope of s(t)

instantaneous velocity $= s'(t)$ "calculus" slope of s(t)

b) Find the *average* velocity of the object from $t = 1$ to $t = 6$.

$$\text{ave. vel.} = \frac{s(6) - s(1)}{6 - 1} = \frac{15}{5} = \boxed{3m/s}$$



c) Find the velocity of the object at $t = 1$, at $t = 2$, and at $t = 4$.

instantaneous

$s = (t-2)^2 - 3$

$s = t^2 - 4t + 4 - 3$

$s = t^2 - 4t + 1$

$v = s' = 2t - 4$

$v(1) = 2(1) - 4 = -2m/s$ (backwards)

$v(2) = 2(2) - 4 = 0m/s$ (stopped)

$v(4) = 2(4) - 4 = 4m/s$ (forward)

d) Find the total distance traveled from $t = 1$ to $t = 6$.

need to know when traveling forward $(2, \infty)$, backward $[0, 2)$, stopped $t=2$

backward: $s(2) - s(1) = (2-2)^2 - 3 - [(1-2)^2 - 3] = 0 - 3 - (1 - 3) = -1m$ (back 1)

forward: $s(6) - s(2) = (6-2)^2 - 3 - [(2-2)^2 - 3] = 16 - 3 - (0 - 3) = 16m$ (forward 16)

Total distance $= 1 + 16 = \boxed{17m}$

1m 16m

Acceleration: rate of change of velocity

average acceleration $= \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}$

instantaneous acceleration $= v'(t) = s''(t)$

e) Find the *average* acceleration of the object from $t = 1$ to $t = 6$. $v = 2t - 4$

ave. accel. $= \dfrac{v(6)-v(1)}{6-1} = \dfrac{2\cdot6-4-[2\cdot1-4]}{5} = \dfrac{12-4-2+4}{5} = \dfrac{10}{5}$

$= \boxed{2\,m/s^2}$

f) Find the acceleration of the object at $t = 5$.

$a(t) = 2$ (acceleration is always $2m/s^2$ for all t)

$\boxed{a(5) = 2m/s^2}$

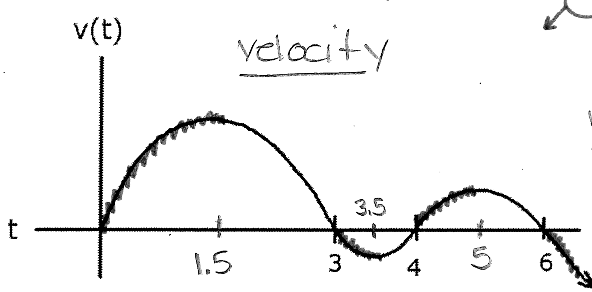
Speed: = |velocity|

speeding up: a & v are same sign

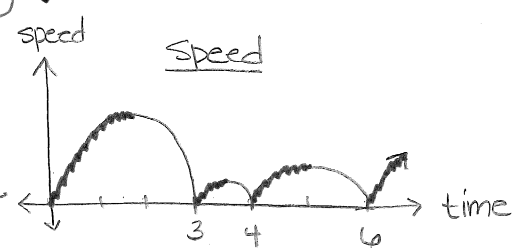
slowing down: a & v are different signs

Example: Describe the motion of the particle.

bold parts = speeding up



reflect graph parts below x-axis above the x-axis



stopped: when $v=0$

$\boxed{@\,t = 0, 3, 4, 6}$

backwards: when $v<0$

$\boxed{(3,4), (6,\infty)}$

forwards: when $v>0$

$(0,3), (4,6)$

speeding up: when a & v same sign

both positive: $(0,1.5), (4,5)$

both negative: $(3,3.5), (6,\infty)$

slowing down: when a & v have diff. sign

$v+, a-$: $(1.5, 3), (5,6)$

$v-, a+$: $(3.5, 4)$

Example: A company estimates the cost in dollars of production for x items is:

$$C(x) = 10{,}000 + 5x + .01x^2$$

a) What is the cost of producing 500 items? 501 items?

$c(500) = 10{,}000 + 5(500) + .01(500)^2 = \$15{,}000$

$c(501) = 10{,}000 + 5(501) + .01(501)^2 = \$15{,}015.01$

ALGEBRA

b) What is the cost of producing the 501st item?

$c(501) - c(500) = \$15{,}015.01 - \$15{,}000 = \boxed{\$15.01}$

* ave. rate of change of cost $= \dfrac{c(501)-c(500)}{501-500} = \dfrac{15.01}{1} = \$15.01$

Marginal cost: prediction of cost of $(n+1)^{th}$ item

given: $c(x) =$ cost

$c'(x) =$ marginal cost

"CALCULUS" (instantaneous) RATE OF CHANGE

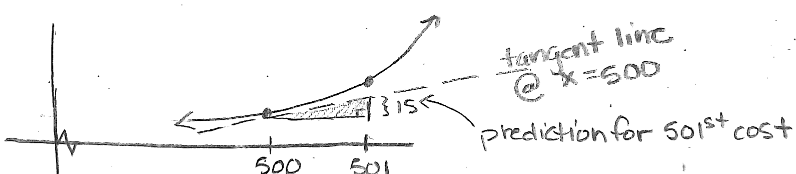
c) What is the marginal cost equation?

$c'(x) = 5 + 2(.01)x^1$

$c'(x) = 5 + .02x$

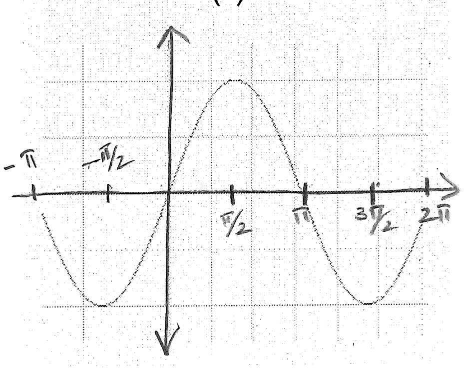
d) What is the marginal cost at the production level of 500 items? (Predicts the cost of the 501st item.)

$c'(500) = 5 + .02(500) = \boxed{\$15.00}$ ← estimate of 501st item based on rate of change at 500th item



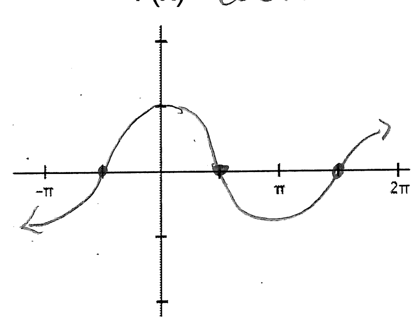
3.5 Derivatives of Trigonometric Functions

f(x) = sinx



X	$\frac{d}{dx}\sin x$
$-\pi/2$	0
$-\pi/2 < x < \pi/2$	+
$\pi/2$	0
$\pi/2 < x < 3\pi/2$	−
$3\pi/2$	0
$3\pi/2 < x < 5\pi/2$	+
$5\pi/2$	0

f'(x) = cos x



WOW!

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\sec x = \sec x \cdot \tan x$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\csc x = -\csc x \cot x$

Example: Find y' if $y = x^2 cosx$

$$y' = x^2(-\sin x) + \cos x(2x)$$

$$\boxed{y' = -x^2\sin x + 2x\cos x}$$

Example: Find the tangent and the normal lines to $y = \frac{tanx}{x}$ at $x = 2$. Support graphically.

$$y' = \frac{x(\sec^2 x) - \tan x(1)}{x^2}$$

$$y(2) = \frac{\tan 2}{2}$$

$$y'(2) = \frac{2\sec^2 2 - \tan 2}{2^2} \approx 3.433$$

slope @ $x=2$

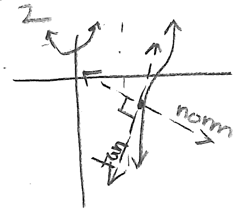
$\left(2, \frac{\tan 2}{2}\right)$ point

$$\frac{-1}{3.433} = -.291$$

tangent line: $y - \frac{\tan 2}{2} = 3.433(x-2)$

normal line: $y - \frac{\tan 2}{2} = -.291(x-2)$

* check by graphing on your calculator:



Example: Find u' and u'' when $u = \frac{\cos x}{1 - \sin x}$.

$$u' = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \overbrace{\sin^2 x + \cos^2 x}^{1}}{(1-\sin x)^2}$$

$$= \frac{-\sin x + 1}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2}$$

$$\boxed{u' = \frac{1}{1-\sin x}}$$

$$u'' = \frac{(1-\sin x)(0) - 1(-\cos x)}{(1-\sin x)^2}$$

$$\boxed{u'' = \frac{\cos x}{(1-\sin x)^2}}$$

Example: The position of an object moving in simple harmonic motion is given by

$$s = 2 + 3\sin t \qquad (s \text{ is in inches and } t \text{ is in seconds})$$

a) Find the displacement of the object from $t = 0$ to $t = \pi$.

$\text{disp} = s(\pi) - s(0)$

$= 2 + 3\sin\pi - (2 + 3\sin 0) = 2 + 3(0) - (2 + 3(0)) = \boxed{0 \text{ inches}}$

b) Describe the motion of the object for $0 \le t \le 8$. $\quad v = s' = 3\cos t$

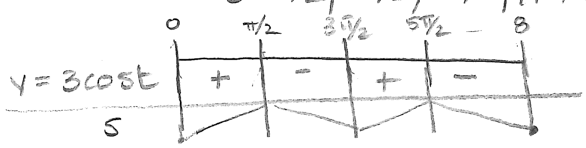
tell when stopped forward backward

stopped:

$v = 0$

$3\cos t = 0$

$t = \pi/2,\ 3\pi/2,\ 5\pi/2, \ldots$ seconds



stopped: $t = \pi/2,\ 3\pi/2,\ 5\pi/2$ seconds

forward: $[0, \pi/2),\ (3\pi/2, 5\pi/2)$

backward: $(\pi/2, 3\pi/2),\ (5\pi/2, 8)$

c) Find the total distance traveled for the object from $t = 0$ to $t = \pi$.

forward $[0, \pi/2)$: $\text{disp} = s(\pi/2) - s(0) = 2 + 3\sin\frac{\pi}{2} - (2 + 3\sin 0) = 2 + 3(1) - (2 + 0)$

$= 3$ inches

backward $(\pi/2, \pi]$: $\text{disp} = s(\pi) - s(\pi/2) = 2 + 3\sin\pi - (2 + 3\sin\frac{\pi}{2})$

$= 2 + 0 - (2 + 3(1)) = -3$ inches

Total distance $= 3 + 3 = \boxed{6 \text{ inches}}$

d) Find the acceleration at $t = \frac{\pi}{2}$.

$a = v' = -3\sin t$

$a(\pi/2) = -3\sin\frac{\pi}{2} = \boxed{-3 \text{ in/s}^2}$

Jerk: rate of change of acceleration

$j(t) = a'(t) = v''(t) = s'''(t)$

e) Find the jerk at $t = \frac{3\pi}{4}$.

$j = a' = -3(\cos t)$

$j(3\pi/4) = -3\cos\frac{3\pi}{4} = -3\left(-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{3\sqrt{2}}{2} \text{ in/s}^3}$

3.6 The Chain Rule

Composite functions: function inside a function, like $f(g(x))$ or $(f \circ g)(x)$

Example: Find $\frac{dy}{dx}$ when $y = (x^3 + 2)^3$

$y = (x^3+2)(x^3+2)(x^3+2) = (x^3+2)(x^6+4x^3+4)$

$= x^9 + 4x^6 + 4x^3 + 2x^6 + 8x^3 + 8$

$y = x^9 + 6x^6 + 12x^3 + 8$

$\boxed{y' = 9x^8 + 36x^5 + 36x^2}$

the long way

Rule 8) Chain Rule: derivative of a composite function

$$\frac{d}{dx} f(g(x)) = \underbrace{f'(g(x))}_{\text{derivative of outside (inside stays sam)}} \cdot \underbrace{g'(x)}_{\text{derivative of inside}}$$

Example: Find $\frac{dy}{dx}$ when $y = (x^3 + 2)^3$...this time, using the chain rule.

Composite function	Outside function	Inside function	Derivative
$f(g(x)) = \sqrt{x^2 - 1}$ $= (x^2-1)^{1/2}$	$\sqrt{\quad}$ or $(\quad)^{1/2}$	$x^2 - 1$	$y' = \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x$ $= \frac{1}{2\sqrt{x^2-1}} \cdot 2x = \frac{x}{\sqrt{x^2-1}}$
$f(g(x)) = (2x^5 + 4x)^7$	$(\quad)^7$	$2x^5 + 4x$	$y' = 7(2x^5+4x)^6 \cdot (10x^4+4)$
$f(g(x)) = \sec^3 x$ $= (\sec x)^3$	$(\quad)^3$	$\sec x$	$y' = 3(\sec x)^2 \cdot \sec x \tan x$ $y' = 3\sec^3 x \tan x$
$f(g(x)) = \sin 3x$ $= \sin(3x)$	$\sin(\quad)$	$3x$	$y' = \cos(3x) \cdot 3$ $y' = 3\cos 3x$

u = function of x $\quad \frac{du}{dx}$ = derivative of u w.r.t. x $\quad$ for chain rule: derivative of inside

$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$	$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$	$\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$	$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$	$\frac{d}{dx}\csc u = -\csc u \cdot \cot u \cdot \frac{du}{dx}$

{MEMORIZE THIS} cross out previous table

Example: Find $\frac{dy}{dx}$

a) $y = \frac{x^2+3}{(2x+1)^2}$

common factor $= (2x+1)\cdot 2$

$$y' = \frac{(2x+1)^2(2x) - (x^2+3)(2)(2x+1)(2)}{((2x+1)^2)^2}$$

$$= \frac{2(2x+1)[(2x+1)\cdot x - (x^2+3)\cdot 2]}{(2x+1)^4}$$

$$= \frac{2[2x^2+x-(2x^2+6)]}{(2x+1)^3} = \boxed{\frac{2(x-6)}{(2x+1)^3}}$$

b) $y = \sec(\tan x)$

$$\boxed{y' = \sec(\tan x)\tan(\tan x)\cdot \sec^2 x}$$

c) $y = \sin^9 13x = (\sin 13x)^9$

$$y' = 9(\sin 13x)^8 \cdot \cos 13x \cdot 13$$

$$\boxed{y' = 117(\sin 13x)^7 \cos 13x}$$

d) $y = \cos^{100}(x^2+3) = [\cos(x^2+3)]^{100}$

$$y' = 100\cos^{99}(x^2+3)\cdot -\sin(x^2+3)\cdot 2x$$

$$y' = -200x\cos^{99}(x^2+3)\sin(x^2+3)$$

3.7 Implicit Differentiation

Example: Find $\frac{dy}{dx}$.

a) $x = \tan y$

y is a function of x

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y))$$

$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

chain rule: (derivative of function y)

$$\frac{1}{\sec^2 y} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \cos^2 y}$$

b) $2xy = 4y + x^3$

L R

$(2x)(y)$

$$2x\cdot\frac{dy}{dx} + y\cdot 2 = 4\cdot\frac{dy}{dx} + 3x^2$$

$$2x\frac{dy}{dx} - 4\frac{dy}{dx} = 3x^2 - 2y \quad \leftarrow \text{get all } \tfrac{dy}{dx} \text{ terms to 1 side}$$

$$\frac{dy}{dx}(2x-4) = 3x^2 - 2y \quad \leftarrow \text{factor out } \tfrac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2-2y}{2x-4}}$$

Example: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y^2 = x^2 + 2x$.

y''

$$2y \cdot \frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{2x+2}{2y} \rightarrow \frac{dy}{dx} = \frac{\cancel{2}(x+1)}{\cancel{2} \cdot y} \rightarrow \boxed{\frac{dy}{dx} = \frac{x+1}{y}}$$

$$\frac{d^2y}{dx^2} = \frac{y(1) - (x+1)\frac{dy}{dx}}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2} = \frac{\frac{y}{y} \cdot y - \frac{(x+1)^2}{y}}{y^2}$$

$$= \frac{\frac{y^2 - (x+1)^2}{y}}{\frac{y^2}{1}} = \frac{y^2 - (x+1)^2}{y} \cdot \frac{1}{y^2} = \boxed{\frac{y^2 - (x+1)^2}{y^3}} = \frac{d^2y}{dx^2}$$

Example: Find the equation of the line tangent to $x^2 + xy - y^2 = 1$ at $(2, 3)$.

product $x \cdot y$

$$2x + x\frac{dy}{dx} + y(1) - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx}(x - 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x - 2y}$$

$$\frac{dy}{dx} @ (2,3): \ m = \frac{-2(2) - 3}{2 - 2(3)} = \frac{-4-3}{2-6} = \frac{-7}{-4} = 7/4$$

$$\boxed{y - 3 = \frac{7}{4}(x - 2)}$$

product: $11x \cdot y^2$

Example: Find the equation of the normal line to $4x^3 + 11xy^2 - 2y^3 = 32$ at $(1, 2)$.

perpendicular to tangent line

$$12x^2 + 11x\left(2y\frac{dy}{dx}\right) + y^2(11) - 6y^2\left(\frac{dy}{dx}\right) = 0$$

$$22xy\left(\frac{dy}{dx}\right) - 6y^2\left(\frac{dy}{dx}\right) = -12x^2 - 11y^2$$

$$\frac{dy}{dx}(22xy - 6y^2) = -12x^2 - 11y^2$$

$$\frac{dy}{dx} = \frac{-12x^2 - 11y^2}{22xy - 6y^2}$$

$$\frac{dy}{dx} @ (1,2): \ m = \frac{-12(1) - 11(2)^2}{22(1)(2) - 6(2)^2} = \frac{-12-44}{44-24} = \frac{-56}{20} = \frac{-14}{5} \perp \frac{5}{14}$$

$$\boxed{y - 2 = \frac{5}{14}(x - 1)}$$

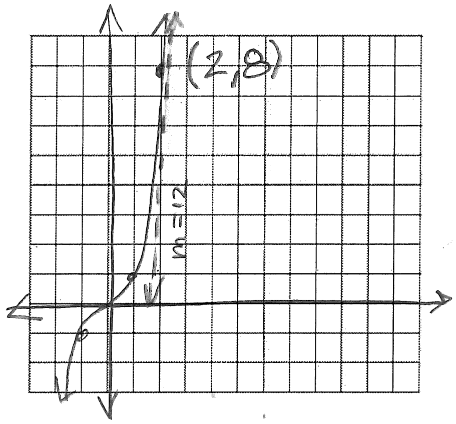
3.8 Derivatives of inverse Trigonometric Functions

Theorem: If f is differentiable and f' is not zero on an interval *I*, then f^{-1} is differentiable on I.

Example: Given: $f(4) = -7$ and $\frac{d}{dx} f(4) = \frac{2}{3} = \frac{\Delta y}{\Delta x}$

Find: $f^{-1}(-7) = 4$ and $\frac{d}{dx} f^{-1}(-7) = \frac{3}{2}$ reciprocal

Example:



$f(x) = x^3$

$f(2) = \underline{8}$

$\frac{d}{dx} f(x) = \underline{3x^2}$

$3(2)^2$

$\frac{d}{dx} f(2) = \underline{12}$

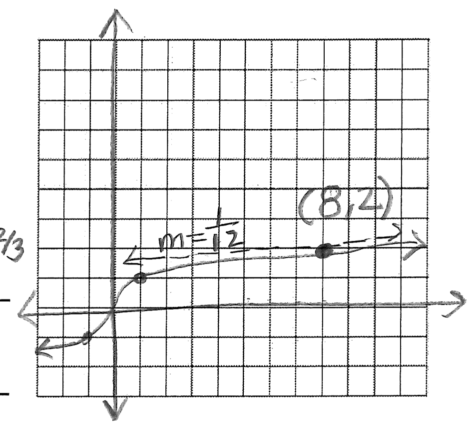
$f^{-1}(x) = \underline{\sqrt[3]{x}}$

$f^{-1}(8) = \underline{2}$

$\frac{d}{dx} f^{-1}(x) = \underline{\frac{1}{3}x^{-2/3}}$

$\frac{d}{dx} f^{-1}(8) = \underline{\frac{1}{12}}$

$\frac{1}{3} \cdot (8^{-2/3}) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 4}$



Example: Find y' when $y = \sin^{-1} x$.

$\sin y = \sin(\sin^{-1} x)$

$\sin y = x$

$\cos y \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$

$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$ derivative of $\sin^{-1} x$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$

$\cos y = \sqrt{1 - x^2}$

Example: Find y' when $y = \sin^{-1}(x^3)$

$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$ need chain rule (deriv. of inside)

$\boxed{y' = \frac{3x^2}{\sqrt{1-x^6}}}$

$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$	$\frac{d}{dx}\tan^{-1}u = \frac{1}{u^2+1}\cdot\frac{du}{dx}$	$\frac{d}{dx}\sec^{-1}u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\cdot\frac{du}{dx}$
$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$	$\frac{d}{dx}\cot^{-1}u = \frac{-1}{u^2+1}\cdot\frac{du}{dx}$	$\frac{d}{dx}\csc^{-1}u = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}}\cdot\frac{du}{dx}$

Example: Find y'.

a) $y = \sin^{-1}(\sqrt{2}t)$

$\sqrt{2}\cdot t$

$$y' = \frac{1}{\sqrt{1-(\sqrt{2}t)^2}}\cdot\sqrt{2} = \frac{\sqrt{2}}{\sqrt{1-2t^2}} = \boxed{\sqrt{\frac{2}{1-2t^2}}}$$

b) $y = \sec^{-1}(t^{-2})$

$\frac{1}{t^2}$

$$y' = \frac{1}{\left|\frac{1}{t^2}\right|\sqrt{\left(\frac{1}{t^2}\right)^2-1}}\cdot -2t^{-3} = \frac{\frac{-2}{t^3}}{\left|\frac{1}{t^2}\right|\sqrt{\frac{1}{t^4}-\frac{t^4}{t^4}}} = \frac{\frac{-2}{t^3}}{\left|\frac{1}{t^2}\right|\sqrt{\frac{1-t^4}{t^4}}}$$

$$= \frac{\frac{-2}{t^3}}{\left|\frac{1}{t^2}\right|\left|\frac{1}{t^2}\right|\sqrt{1-t^4}} = \frac{\frac{-2}{t^3}}{\left|\frac{1}{t^4}\right|\sqrt{1-t^4}} = \frac{-2}{t^3}\cdot\frac{t^4}{\sqrt{1-t^4}} = \frac{-2t}{\sqrt{1-t^4}}$$

$\sqrt{t^2} = |t|$

$|t^4| = t^4$

c) $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1}x,\ x > 1$ → so $|x| = x$

$$y' = \frac{1}{(\sqrt{x^2-1})^2+1}\cdot\frac{1}{2}(x^2-1)^{-1/2}\cdot 2x + \frac{-1}{|x|\sqrt{x^2-1}} = \frac{2x}{x^2-1+1}\cdot\frac{1}{2(x^2-1)^{1/2}} + \frac{-1}{x\sqrt{x^2-1}}$$

$$= \frac{x}{x^2}\cdot\frac{1}{\sqrt{x^2-1}} + \frac{-1}{x\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}} + \frac{-1}{x\sqrt{x^2-1}} = \boxed{0}$$

wow!

d) Write the equation of the line tangent to $y = \tan^{-1}x$ at $(1, \frac{\pi}{4})$.

$$y' = \frac{1}{x^2+1}$$

$$y' @ (1, \pi/4):\ \frac{1}{1^2+1} = \frac{1}{2}$$

Slope

$$\boxed{y - \frac{\pi}{4} = \frac{1}{2}(x-1)}$$

3.9 Derivatives of Exponentials and Logarithms

special case: $e^u \cdot \ln e \cdot \frac{du}{dx}$

$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$	$\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx}$
$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$ $a > 0,\ a \neq 1$	$\frac{d}{dx}\log_a u = \frac{1}{u \ln a} \cdot \frac{du}{dx}$

special case: $\frac{1}{u \ln e} \cdot \frac{du}{dx}$

Example: Find the derivative of the following.

a) $y = e^{3x}$

$y' = e^{3x} \cdot \frac{d}{dx}(3x) = e^{3x} \cdot 3$

$\boxed{y' = 3e^{3x}}$

b) $y = e^{x^2-5x}$

$y' = e^{x^2-5x} \cdot \frac{d}{dx}(x^2-5x)$

$\boxed{y' = (2x-5)e^{x^2-5x}}$

c) $y = 3^{4x+1}$

$y' = 3^{4x+1} \cdot \ln 3 \cdot \frac{d}{dx}(4x+1)$

$\boxed{y' = 4\ln 3 \cdot 3^{4x+1}}$

d) $y = \log\sqrt{2x+1} = \log_{10}(2x+1)^{1/2}$

$y' = \frac{1}{(2x+1)^{1/2}\ln 10} \cdot \frac{d}{dx}(2x+1)^{1/2}$

$= \frac{1}{(2x+1)^{1/2}\ln 10} \cdot \frac{1}{2}(2x+1)^{-1/2}(2)$

$= \frac{1}{(2x+1)^{1/2}\ln 10} \cdot \frac{1}{(2x+1)^{1/2}} = \boxed{\frac{1}{(2x+1)\ln 10}}$

Example: Find the equation of the line tangent to $y = \ln x^2$ at $x = e$.

point: $(e, f(e))$ $\quad y(e) = \ln(e^2) = 2 \cdot \ln e = 2$

$(e, 2)$

slope: $y' = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$

@ $x = e$: $m = \frac{2}{e}$

$\boxed{y - 2 = \frac{2}{e}(x - e)}$

Example: At what point does the tangent line for $y = 2^t - 3$ have a slope of 21?

$y' = 2^t \ln 2$

$$2^t \ln 2 = 21$$
$$2^t = \frac{21}{\ln 2}$$
$$\ln(2^t) = \ln\left(\frac{21}{\ln 2}\right)$$
$$t \cdot \ln 2 = \ln\left(\frac{21}{\ln 2}\right)$$
$$t = \frac{\ln\left(\frac{21}{\ln 2}\right)}{\ln 2} \approx 4.921$$

$$y(4.921) \approx 2^{4.921} - 3 \approx 27.297$$

use stored value

$$\boxed{(4.921,\ 27.297)}$$

Example: Find y' if $y = \sqrt{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}} = \left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/2}$

skip #46 in HW if we don't have time for this example

$$\ln y = \frac{1}{2} \cdot \ln\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)$$
$$\ln y = \frac{1}{2}\left[\ln(x-3)^4 + \ln(x^2+1) - \ln(2x+5)^3\right]$$
$$\ln y = \frac{4}{2}\ln(x-3) + \frac{1}{2}\ln(x^2+1) - \frac{3}{2}\ln(2x+5)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{(x-3)} + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot 2x - \frac{3}{2} \cdot \frac{1}{2x+5} \cdot 2$$
$$\frac{dy}{dx} = \left(\frac{2}{x-3} + \frac{1}{x^2+1} - \frac{3}{2x+5}\right) \cdot \frac{y}{1}$$
$$\boxed{\frac{dy}{dx} = \left(\frac{2}{x-3} + \frac{1}{x^2+1} - \frac{3}{2x+5}\right) \cdot \sqrt{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}}}$$